On Time Inconsistency: A Technical Issue in Stackelberg Differential Games

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Abstract

Stackelberg differential games are useful settings in which optimal government policies can be studied. This paper argues that the analysis of these games involves a key technical issue. In particular, we question the necessity for optimality of one boundary condition invoked in existing literature. The issue is of key interest because the boundary condition is largely responsible for the time inconsistency results previously obtained. We show that the boundary condition is not necessary in some cases. As a result, our finding undermines the credibility of the existing conclusions. *Journal of Economic Literature Classification Numbers*: C61, E62, H21.

1 Introduction

Stackelberg differential games have often been used to study dynamic interaction between government and private agents. The government naturally plays the role of the leader, setting monetary and fiscal policies. Private agents are the followers, responding optimally to government policy in their decision on consumption, investment, labor supply and so on. The government then takes the private agents' best response into account and forms the optimal policy. Examples using such a framework can be found in Kydland and Prescott [10], Calvo [4], Turnovsky and Brock [15], Lucas and Stokey [11], Chamley [7], and Persson, Persson and Svensson [13].

The pioneering work by Kydland and Prescott [10] has created the "time inconsistency" literature, which generally falls into two categories. In the first category, the studies determine which optimal government policies tend to be time inconsistent. Calvo [4], Turnovsky and Brock [15] and Chamley [7] are examples in this category.

In the second category, the studies examine the debt instruments the incumbent government can use to bind the action of its successor and thus ensure time consistency. Lucas and Stokey [11] and Persson, Persson and Svensson [13] are representative.

The work in the second category carries an optimistic tone that the time inconsistent optimal policy can be made time consistent through intentional debt management by governments. A technical error found in Persson, Persson and Svensson [13] shatters such a hope (see Calvo and Obstfeld [6]). Recent attempts in characterizing time consistent policies without commitment include Calvo and Guidotti [5], Benhabib and Rustichini [1] and Benhabib, Rustichini and Velasco [2]. Calvo and Guidotti show that a second-best incentive compatible outcome can be reached by manipulating government debt maturity. Benhabib et al. show that incentive compatible outcomes can be maintained through reputational mechanisms and they conduct numerical analysis to see how differences in taxes with and without commitment are affected by parameter changes.

The work in the first category also suffers from technical inadequacy as this paper will argue. Unless the issue raised here is resolved, the results obtained in a number of existing papers remain questionable.

To put it simply, the technical inadequacy lies in a failure to prove that the boundary conditions imposed in various papers are necessary for optimality. In particular, we show that one of the boundary conditions is not necessary in some cases. Since this boundary condition is responsible for deriving time inconsistency, our finding calls for caution.

This paper only points out a problem but does not offer a solution. We hope that this paper serves the same purpose as the counter-example in Shell [14]: The Shell example questions the necessity of a transversality condition for optimality in infinite horizon and has served as a challenge which leads to the full resolution of the problem by Weitzman [16] in discrete time and Benveniste and Scheinkman [3] in continuous time.

The rest of the paper is organized as follows. Section 2 presents a dynamic taxation model as a Stackelberg differential game. Section 3 uses the boundary conditions that are usually imposed in the literature and obtain the time inconsistency result. Section 4 uses a special parametric example of the model which permits explicit solution to show that one of the boundary condition imposed in Section 3 is not necessary for optimality. Thus the conclusion of time inconsistency reached in Section 3 is spurious. Section 5 discusses the generality of our finding. Section 6 contains an example which shows that the technical issue can arise in the original Chamley framework. Section 7 concludes.

2 A Model of Dynamic Taxation

We consider an economy populated with a continuum of identical private agents. These agents are consumers-cum-producers and can be represented by a set $\{\alpha : \alpha \in [0,1]\}$. They produce a single good which is either consumed or invested as capital for later production. Production function for an individual is given by $y = f(k)$ where k is capital and y is output.

Government imposes only an output tax, the entire path of which, $\{\tau(t) \in [0,1]\colon t \geq 0\}$, is announced at time zero. Government spending is in the form of public goods. We require that the government budget be balanced each period. In other words,

$$g(t) = \int_0^1 f(k_\alpha(t))\tau(t)\, d\alpha, \text{ for all } t \geq 0, \tag{1}$$

where g is the amount of public good, k_α is the capital stock of individual α.

For an individual α, his optimal consumption and investment plan is the solution to the following lifetime utility maximization problem:
(Problem α)

$$\max \int_0^\infty [U(c_\alpha) + V(g)]\ e^{-\rho t}\ dt, \tag{2}$$

$$\text{subject to } \dot{k}_\alpha = f(k_\alpha)(1-\tau) - c_\alpha, \text{ with } k_{\alpha 0} \text{ given,} \tag{3}$$

where c_α is individual α's consumption; the utility functions, $U(\cdot)$ and $V(\cdot)$, and the production function, $f(\cdot)$, are concave and strictly increasing. Furthermore, $U'(0) = V'(0) = +\infty$.

We can characterize the response of individual α to government tax policy by a set of first order conditions and a transversality condition. To do this, we use the Hamiltonian:

$$H_\alpha = [U(c_\alpha) + V(g)] + q_\alpha \left[f(k_\alpha)(1-\tau) - c_\alpha\right]. \tag{4}$$

Note that g depends on the action of the entire population (see Eq. (1)) and is thus considered as given by any individual. Therefore, the first order conditions are:

$$U'(c_\alpha) = q_\alpha \tag{5}$$

$$\dot{q}_\alpha = \rho q_\alpha - q_\alpha f'(k_\alpha)(1-\tau). \tag{6}$$

The transversality condition is $k_\alpha q_\alpha e^{-\rho t} \to 0$ as $t \to \infty$.

For each path $\{\tau(t) \in [0,1]\colon\ t \geq 0\}$ that the government announces, there is a corresponding optimal plan of resource allocation over time by each individual. Since all individuals are assumed to be identical, the subscript α can be removed from the equations above and the optimal response of private agents can thus be rewritten as follows:

$$U'(c) = q \tag{7}$$

$$\dot{k} = f(k)(1-\tau) - c \tag{8}$$

$$\dot{q} = \rho q - q f'(k)(1-\tau) \tag{9}$$

$$k_0 \text{ given and } \lim_{t\to\infty} kqe^{-\rho t} = 0. \tag{10}$$

In equilibrium, Eq. (1) reduces to

$$g = f(k)\tau. \tag{11}$$

Assume the government shares the same objective function with the representative individual. If we invert Eq. (7) to obtain $c = c(q)$, we can write down the government's optimization problem as follows:

(Problem G)

$$\max \int_0^\infty [U(c(q)) + V(f(k)\tau)] \ e^{-\rho t} \ dt,$$

$$\text{subject to: } \dot{k} = f(k)(1-\tau) - c(q) \tag{12}$$

$$\dot{q} = \rho q - q f'(k)(1-\tau) \tag{13}$$

$$k_0 \text{ given and } \lim_{t\to\infty} kqe^{-\rho t} = 0 \tag{14}$$

This maximization problem has some peculiar aspects. First, the objective function may not be concave in q. Second, the boundary conditions include a transversality condition at infinity. As a result, there are no ready-made necessary boundary conditions we can apply for (Problem G).

The first order conditions should still be straightforward to derive. Let λ and ξ be the co-state variables for k and q, respectively. We have:

$$V'(f(k)\tau)f(k) = \lambda f(k) - \xi q f'(k) \tag{15}$$

$$\dot{\lambda} = \rho\lambda - V'(f(k)\tau)f'(k)\tau - \lambda f'(k)(1-\tau) + \xi q f''(k)(1-\tau) \tag{16}$$

$$\dot{\xi} = \rho\xi - qc'(q) + \lambda c'(q) - \xi(\rho - f'(k)(1-\tau)) \tag{17}$$

In the next section, we will see how the boundary conditions conventionally imposed imply time inconsistency of optimal government tax policy.

3 Necessary Boundary Conditions

As mentioned in the last section, (Problem G) is not a standard maximization problem. The necessity of the boundary conditions ought to be established rigorously. The special aspects of (Problem G) have not received any attention and researchers simply apply their intuition when selecting the boundary conditions.

If we follow Turnovsky and Brock [15], we should impose two transversality conditions: $k\lambda e^{-\rho t} \to 0$ and $q\xi e^{-\rho t} \to 0$. Chamley [7, 8] did not impose similar conditions explicitly, but since convergence to steady state is assumed, he may have imposed the transversality conditions implicitly. If we follow Chamley, we should also impose a third condition: $\xi_0 = 0$. The rationale given in Chamley [7, 8] for imposing $\xi_0 = 0$ is that q_0 seems free to move. Persson and Svensson [12] uses this same boundary condition based on the same rationale.

With $\xi_0 = 0$ imposed as a necessary condition for optimality, the optimal government policy must be time inconsistent. We put this in a proposition.

PROPOSITION 1: *If $\xi_0 = 0$ is necessary for optimality, the optimal government policy is time inconsistent.*

PROOF: Suppose the optimal government policy is time consistent. We proceed in finding a contradiction. In fact time consistency and the necessity of $\xi_0 = 0$ imply that the solution to (Problem G) must have the property that $\xi(t) \equiv 0$ for all t. Eq. (17) then implies that $\lambda(t) \equiv q(t)$ for all t. With this result in mind, we see that equations (13) and (16) imply $V'(f(k)\tau)f'(k)\tau \equiv 0$. Since $V(\cdot)$ and $f(\cdot)$ are strictly increasing, it must be the case that $\tau \equiv 0$. The zero tax policy however cannot be optimal because $V'(0) = +\infty$. We have found a contradiction. Thus the optimal government policy is time inconsistent.

REMARK: The time inconsistency results obtained in Chamley [7, 8] and Persson and Svensson [12] critically depend on the necessity of a boundary condition equivalent to $\xi_0 = 0$. In the next section, we will use a parametric version of our model to show that $\xi_0 = 0$ is not necessary in some cases.

4 A Counter Example

In the last section, we showed that the conventional use of the boundary condition $\xi_0 = 0$ leads to the straightforward conclusion of time inconsistency. We will show in this section that imposing $\xi_0 = 0$ as a necessary condition for optimality is spurious.

The parametric class of examples we have is: $U(c) = [c^{1-\sigma} - 1]/(1-\sigma)$ and $f(k) = Ak^{\sigma}$ where $0 < \sigma \leq 1$ and $A > 0$[1]. The simplest case in which $\sigma = 1$ is sufficient to demonstrate our argument. In this case, our representative individual has log utility and linear production:

$$U(c) = \ln(c),\ V(g) = \ln(g),\ f(k) = Ak.$$

The private agents' behavior is now characterized by:

$$1/c = q \tag{18}$$

$$\dot{k} = Ak(1-\tau) - c \tag{19}$$

$$\dot{q} = \rho q - qA(1-\tau) \tag{20}$$

$$k_0 \text{ given, and } kqe^{-\rho t} \to 0. \tag{21}$$

Eq. (18) implies that $c(q) = 1/q$. Substitute this into (19) and multiply through the equation by q. Add $\dot{q}k$ to each side, then substitute out $\dot{q}$ on the RHS from (20). This leaves

$$d\,(qk)\,/dt = \rho qk - 1.$$

[1] This particular class of utility-production pairs, indexed by σ, can often be used to obtain closed form solutions in dynamic settings. See Xie [17, 18] for similar examples.

The solution is

$$qk = \frac{1}{\rho} + Je^{\rho t},$$

with J an arbitrary constant. However, the transversality condition then implies that $J = 0$. Hence we have:

$$q = 1/(\rho k) \tag{22}$$

Because of this link between k and q, there are now two ways to solve (Problem G). First, we can substitute $q = 1/(\rho k)$ into (Problem G) so that it becomes a concave optimal control problem with a single state variable k and a single control variable τ:

$$\text{Max} \int_0^\infty [\ln(\rho k) + \ln(Ak\tau)]e^{-\rho t}dt,$$

$$\text{subject to } \dot{k} = Ak(1-\tau) - \rho k, \text{ and } k_0 \text{ given.}$$

The solution to this optimization problem surely exists, and it is given explicitly as follows:

$$\tau = \frac{\rho}{2A} \text{ and } k(t) = k_0 e^{(A-3\rho/2)t}. \tag{23}$$

Alternatively, if the substitution $q = 1/(\rho k)$ is not made, (Problem G) is an optimal control problem with two state variables, k and q, and one control variable τ as described in the last section. The corresponding first order conditions are:

$$\tau = 1/[(k\lambda - \xi q)A] \tag{24}$$

$$\dot{k} = Ak(1-\tau) - 1/q \tag{25}$$

$$\dot{q} = \rho q - qA(1-\tau) \tag{26}$$

$$\dot{\lambda} = \rho\lambda - 1/k - \lambda A(1-\tau) \tag{27}$$

$$\dot{\xi} = \rho\xi + 1/q - \lambda/q^2 - \xi(\rho - A(1-\tau)) \tag{28}$$

The known necessary boundary conditions are: k_0 given, and $kqe^{-\rho t} \to 0$, as $t \to \infty$. What are the other necessary boundary conditions for optimality? In particular, is the condition $\xi_0 = 0$ really necessary?

The merit of this example is that one can actually solve the differential equations above (as in Appendix A) for the general solution. The resulting tax rate can be shown to be

$$\tau = \frac{\rho}{(2 - \Omega\rho e^{\rho t})A} \tag{29}$$

where Ω denotes $\lim_{t\to\infty} q\xi e^{-\rho t}$.

The two approaches to solving (Problem G) must yield the same solution. This requires $\Omega = \lim_{t\to\infty} q\xi e^{-\rho t} = 0$ by comparing Eq. (29) with (23). The condition $\xi_0 = 0$ is clearly not necessary. As a result, the conventional argument to explain why optimal government policies are time inconsistent is incorrect in this example. It is easily verified that the optimal tax policy, $\tau = \rho/(2A)$, is in fact time consistent.

The intuition which explains why the condition $\xi_0 = 0$ is not necessary for optimality is as follows. In our example, since we have $q = 1/(\rho k)$, q_0 equals $1/(\rho k_0)$ and is thus independent of the tax policy. In other words, q_0 is non-controllable by $\{\tau(t) \in [0,1]:\ t \geq 0\}$. The condition $\xi_0 = 0$ is necessary only if q_0 is controllable in the sense that q_0 can be moved around by changing the tax policy.

What happens in a more general case when $\sigma \in (0,1)$? Is q_0 still non-controllable? We have the following proposition.

Proposition 2. When $0 < \sigma < 1$, the private agents' best response to the government tax policy has the property that $q = [\sigma/(\rho k)]^{\sigma}$. Therefore, q_0 is determined by k_0 and is non-controllable by government tax policies.

Proof: See Appendix B.

Remark: Proposition 2 indicates that the conclusion in the special case when $\sigma = 1$ carries through to the more general case when $\sigma \in (0, 1)$. Thus for any $\sigma \in (0, 1]$, q_0 is non-controllable. As a result, $\xi_0 = 0$ is not necessary and the optimal tax policy is time consistent.

Note that when $\sigma \in (0, 1)$, both the utility function and the production function are well behaved and satisfy the usual assumptions. Therefore, it is difficult to rule the class of counter examples out. With general utility and production functions, explicit solution is impossible. Thus studies that allow us to identify a priori whether q_0 is controllable can help us understand time inconsistency.

5 Generality of Our Result

In the last section we showed that $q_0 = 0$ is not necessary for optimality for a class of utility-production pairs indexed by σ. This is because with these pairs, q_0 is non-controllable. There may be other utility-production pairs with the same property. Time inconsistency result can only be trusted when these pairs are excluded. Therefore further studies are needed to identify the cases for exclusion. The following proposition is a start.

Proposition 3. Let $U(\cdot)$ and $f(\cdot)$ be concave and strictly increasing with $f(0) = 0$. If

$$f(k)U'[\rho f(k)/f'(k)] \equiv J, \text{ a constant,} \tag{30}$$

then q_0 is non-controllable.

PROOF: We will show that $c(t) = \rho f(k(t))/f'(k(t))$ is private agents' optimal choice for consumption regardless of $\{\tau(t) \in [0,1]\colon\ t \geq 0\}$. Hence $q_0 = U'(c(0)) = J/f(k_0)$ is non-controllable.

Differentiate Eq. (30) with respect to k. The result can then be used to verify that the consumption choice above satisfies all the first order conditions (7) to (9). It thus remains to show that the transversality condition is also satisfied. Note that $q(t) = U'(c(t)) = J/f(k(t))$. All we need to show is $\lim_{t\to\infty} Jk(t)e^{-\rho t}/f(k(t)) = 0$. Eq. (8) implies that $\dot{k} \geq -\rho f(k)/f'(k)$. Thus

$$f(k(t)) \geq f(k_0)e^{-\rho t} \text{ when } k_0 > 0.$$

Since $f(\cdot)$ is strictly increasing, the above inequality implies that $k(t) > 0$ for all t. Let $z(t)$ denote $k(t)/f(k(t))$. Then we have $z(t) > 0$ for all t and:

$$\begin{aligned}
\dot{z} &= \dot{k}\left[1 - f'(k)k/f(k)\right]/f(k) \\
&= \left[f(k)(1-\tau) - \rho f(k)/f'(k)\right]\left[1 - f'(k)k/f(k)\right]/f(k) \\
&\leq \left[1 - f'(k)k/f(k)\right] \\
&\leq 1
\end{aligned}$$

where the first inequality is derived from three facts: (i) $\tau \in [0,1]$; (ii) $k(t) > 0$; (iii) $1 - f'(k)k/f(k) \geq 0$ due to the concavity of $f(\cdot)$ and $f(0) = 0$. Thus we obtain $0 < z(t) \leq z(0) + t$, which implies that $\lim_{t\to\infty} kqe^{-\rho t} = \lim_{t\to\infty} Jk(t)e^{-\rho t}/f(k(t)) = \lim_{t\to\infty} ze^{-\rho t} = 0$. ■

REMARK: The above proof is fast paced. Note that for $U(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ and $f(k) = Ak^{\sigma}$, Eq. (30) holds. Thus Proposition 2 is a special case of Proposition 3. In Appendix B, Proposition 2 is shown in a leisurely style for easy grasp.

In the case when q_0 *is* controllable, $\xi_0 = 0$ is necessary for optimality and time inconsistency occurs. In order to characterize the optimal policy so that it may be compared with the sub-optimal ones, we need to know about other necessary boundary conditions. In particular, are $\lim_{t\to\infty} k\lambda e^{-\rho t} = 0$ and $\lim_{t\to\infty} q\xi e^{-\rho t} = 0$ necessary for optimality? If they are, then we have five boundary conditions on a system of four differential equations. This suggests the possibility that the system is over-determined.

In our special example when $\sigma = 1$, we find that $\lim_{t\to\infty} q\xi e^{-\rho t} = 0$ is a necessary condition for optimality. On the other hand, $\lim_{t\to\infty} k\lambda e^{-\rho t} = 0$ is automatically satisfied when $\lim_{t\to\infty} kqe^{-\rho t} = 0$ is imposed (Appendix A shows that $k\lambda = B - t$ with B a constant). Will this always happen and therefore we need not explicitly impose $\lim_{t\to\infty} k\lambda e^{-\rho t} = 0$ as a boundary condition?

To summarize, current understanding on Stackelberg differential games is limited. Different authors have different ideas about which boundary conditions are necessary for optimality. Turnovsky and Brock [15] did not impose $\xi_0 = 0$ ($q_3(0) = 0$ in their notation) whereas Chamley [7, 8] and Persson and Svensson [12] did. We definitely need to establish rigorously which boundary conditions are necessary for optimality and when.

6 The Chamley Framework

To be sure, the optimal taxation model studied thus far is different from that in the original Chamley [8]. Our model has public goods in the utility function and assumes no government bonds. While this model is interesting in its own right, it would be more convincing if we can present a counter-example in Chamley's own framework showing that the conclusion of time-inconsistency can be spurious for some specification of functional forms. We are also interested to see whether Chamley's result that asymptotic capital incomes taxes are zero is still correct.

We assume that the representative agent has the following preferences[2]:

$$\int_0^\infty e^{-\rho t} \ln\left[c - l\right] \, dt \tag{31}$$

where c is consumption and l is the effort put into work.

The representative firm has a Cobb-Douglas production function,

$$y = Ak^\alpha l^{1-\alpha} \tag{32}$$

where k is the capital stock.

Given real interest rate r and the real wage w, the firm maximizes its profit by choosing k and l according to the following equations:

$$r = \alpha A k^{\alpha-1} l^{1-\alpha} \tag{33}$$

$$w = (1-\alpha) A k^\alpha l^{-\alpha} \tag{34}$$

As before, we first study the private agent's utility maximization problem and then ask what tax policies the government should adopt. Define a as an individual's total wealth, $a = k + b$, where b is the government bonds that this individual holds. For a given series of $\{\tau_k(t),\ \tau_l(t)\}_0^\infty$, the representative agent maximizes (31) subject to

$$\dot{a} = r(1-\tau_k)a + w(1-\tau_l)l - c \tag{35}$$

with $a_0 = k_0 + b_0$ given.

Let q be the co-state variable associated with a. The first order conditions are:

$$\frac{1}{c-l} = q \tag{36}$$

[2]The preferences structure is borrowed from Greenwood, Hercowitz, and Huffman [9]. They use $u(c,l) = \left[c - \frac{l^{1+\theta}}{1+\theta}\right]^{1-\gamma} / (1-\gamma)$. We simply take the special case: $\gamma = 1$ and $\theta = 0$.

$$\frac{1}{c-l} = qw(1-\tau_l) \tag{37}$$

$$\dot{q} = \rho q - qr(1-\tau_k) \tag{38}$$

The initial condition and the transversality condition is

$$a_0 \text{ given and } aqe^{-\rho t} \to 0 \tag{39}$$

Equations (36) and (37) imply:

$$w(1-\tau_l) = 1 \tag{40}$$

and

$$c = l + \frac{1}{q} \tag{41}$$

Substituting (40) and (41) into the wealth accumulation equation, we obtain,

$$\dot{a} = r(1-\tau_k)a - \frac{1}{q}$$

This equation, combined with (38), yields,

$$\frac{1}{aq}\frac{d(aq)}{dt} = \rho - \frac{1}{aq}$$

With the transversality condition (39) imposed, this differential equation has a unique solution,

$$aq = \frac{1}{\rho} \tag{42}$$

This tight relationship between a and q once again demonstrates that controllability problem can also arise in the original Chamley framework. If we followed what Chamley does, namely imposing $\xi_0 = 0$ (ξ is the co-state variable on q) as a necessary condition in the government problem, we would immediately obtain time-inconsistency. This conclusion of time-inconsistency is however spurious because now we know that whenever controllability problem arises, $\xi_0 = 0$ is not necessary.

To see whether Chamley's result that asymptotic capital incomes taxes are zero still holds, let us continue with the example and find out more about the private agent's decisions.

Equations (41) and (42) imply that

$$c - l = \rho a \tag{43}$$

To see how labor is determined in equilibrium, rewrite Eq. (40) as follows,

$$(1-\alpha)Ak^{\alpha}l^{-\alpha}(1-\tau_l) = 1$$

Therefore, we have:

$$l = [(1-\alpha)A(1-\tau_l)]^{1/\alpha}\, k$$

and as a result,

$$\begin{aligned} rk &= \alpha A k^{\alpha-1} l^{1-\alpha} k \\ &= \alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} k \end{aligned}$$

$$\begin{aligned} wl &= (1-\alpha)Ak^{\alpha}l^{-\alpha} \\ &= (1-\alpha)A\left[(1-\alpha)A(1-\tau_l)\right]^{/\alpha} k \end{aligned}$$

Because of the tight relationship between a and q and the fact that $c-l = \rho a$, the government problem is clearly the following:

$$\max \int_0^{\infty} e^{-\rho t} \ln(\rho a)\, dt$$

$$\begin{aligned} \text{subject to } \dot{a} &= (1-\tau_k)\alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} a - \rho a \\ \dot{b} &= g - \tau_k \alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} a \\ &\quad -\tau_l(1-\alpha)A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} (a-b) \\ &\quad +\alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} b \\ \tau_k &\leq 1 \\ &\quad a_0 \text{ and } b_0 \text{ given} \end{aligned}$$

Where g is the exogenous government spending and is assumed to be constant for simplicity[3]. We show in Appendix C that the optimal taxation policy is:

$$\begin{aligned}\tau_k &= \begin{cases} 1 \text{ when } t < T \\ 0 \text{ when } t \geq T \end{cases} \\ \tau_l &= 0 \text{ for all } t\end{aligned}$$

where T is determined in the following equation:

$$\frac{ra_0}{\rho + r}\left[1 - e^{-(\rho+r)T}\right] = b_0 + \frac{g}{r} \tag{44}$$

where r is the pre-tax real interest rate $\alpha A\left[(1-\alpha)A\right]^{(1-\alpha)/\alpha}$.

In order for the solution T to be finite, we need

$$\frac{ra_0}{\rho + r} > b_0 + \frac{g}{r}$$

Eq. (44) is intuitive. It says that the higher the government spending, the longer the regime of 100% capital income taxation will have to last. Also, the more debt the government inherits at time zero, the longer the regime of 100% capital income taxation will have to last.

To summarize, Chamley's result that τ_k is asymptotically zero still holds in our special example. But his claim that after time T, the government spending is financed by labor income tax seems wrong[4]. This claim also provides the intuition for his time inconsistency result stated in his concluding paragraph that a government may be tempted to raise revenues by future

[3] If g grows at a rate lower than the real interest rate, similar analysis can be done. We need to find a T so that $b(T)$ is sufficiently negative and b grows at the same rate as the government spending for $t \geq T$.

[4] Chamley [8] does not have any discription of the optimal wage tax but does claim that the government spending after time T is financed only by taxing wage income (page 617).

levies on capital. Our special example illustrates that in some cases q_0 is non-controllable and thus $\xi_0 = 0$ is not a necessary condition for optimality in the government problem. This destroys the time inconsistency result technically. Not only that, the example also destroys time inconsistency result intuitively. After time T, the government owns enough assets (b sufficiently negative) so that the government spending can be entirely financed by its interest earnings. The government has no incentive to lengthen the regime of 100% capital income taxation. The length of this regime, T, is chosen optimally once and for all.

7 Conclusion

Due to the nature and complexity of Stackelberg differential games, which boundary conditions are necessary for optimality is an unresolved issue. This paper shows that the current treatment of the issue is not rigorous and conclusion of time inconsistency could be spurious in some cases.

Since the question of the necessity of boundary conditions is directly related to the time inconsistency result, we must find the answer. This paper puts the question forward in a series of examples. As in Shell [14], these examples are very special and are unlikely to be realistic. This does not mean that we can afford to neglect them. In fact, powerful counter-examples are always special and unexpected. We hope that our counter-examples will attract effort for the early resolution of the technical issue in Stackelberg differential games.

Appendix A

Here we derive the general solution for the set of differential equations (25) to (28) with the known necessary boundary conditions: k_0 given and $\lim_{t\to\infty} kqe^{-\rho t} = 0$

As shown in the main text, equations (25), (26) and the transversality condition $\lim_{t\to\infty} kqe^{-\rho t} = 0$ imply that $q = 1/(\rho k)$. Substitute $q = 1/(\rho k)$ into (25). Equations (25) and (27) can then be re-arranged to obtain $d(k\lambda)/dt = -1$. Thus $k\lambda = B - t$,where B is any constant. Hence, we can substitute $\lambda = (B-t)/k = (B-t)\rho q$ into (28) and solve (26) and (28) for ξq. The result is $\xi q = \Omega e^{\rho t} - 2/\rho + (B-t)$, where $\Omega = \lim_{t\to\infty} \xi q e^{-\rho t}$. Therefore, we have

$$\tau = \frac{1}{(k\lambda - \xi q)A} = \frac{\rho}{(2 - \Omega\rho e^{\rho t})A},$$

which gives the general solution depending on the choice of the value for Ω.

Appendix B

In this appendix, we prove Proposition 2 using three lemmas.

When $\sigma \in (0,1)$, the first order conditions characterizing the private agents' behavior are as follows:

$$c^{-\sigma} = q \tag{45}$$

$$\dot{k} = Ak^{\sigma}(1-\tau) - c \tag{46}$$

$$\dot{q} = \rho q - \sigma q A k^{\sigma-1}(1-\tau) \tag{47}$$

Since the private agents' maximization problem is a standard one, the necessary boundary conditions are well established and they are: k_0 given and $\lim_{t\to\infty} kqe^{-\rho t} = 0$.

LEMMA 1: *The general solution to equations* (45) *to* (47) *has the property that* $q^{1/\sigma}k = \sigma/\rho + Je^{\rho t/\sigma}$, *with* J *any constant.*

PROOF: Let $x = q^{1/\sigma}k$. Then we have:

$$\begin{aligned} \dot{x}/x &= (1/\sigma)\dot{q}/q + \dot{k}/k \\ &= \rho/\sigma - 1/x \end{aligned}$$

Thus, $\dot{x} = \rho x/\sigma - 1$. This differential equation has the general solution: $x = \sigma/\rho + Je^{\rho t/\sigma}$, with J any constant. ■

REMARK: The private agents' maximization problem is a standard concave optimal control problem and therefore has a unique solution. Thus in order to show $q = [\sigma/(\rho k)]^{\sigma}$, it remains to show that the transversality condition $\lim_{t\to\infty} kqe^{-\rho t} = 0$ is satisfied when $J = 0$.

LEMMA 2: *When $J = 0$, $k \geq 0$ and is bounded from above.*

PROOF: When $J = 0$, $q^{1/\sigma} k = \sigma/\rho$. Thus, $c = \rho k/\sigma$. Eq. (46) implies that

$$-\rho k/\sigma \leq \dot{k} \leq A k^{\sigma} - \rho k/\sigma.$$

Since $\sigma < 1$, we know that $k \geq 0$ and is bounded from above. ■

LEMMA 3: *When $J = 0$, $\lim_{t\to\infty} kqe^{-\rho t} = 0$.*

PROOF: When $J = 0$, $q^{1/\sigma} k = \sigma/\rho$. We have:

$$\begin{aligned}
\lim\nolimits_{t\to\infty} kqe^{-\rho t} &= \lim\nolimits_{t\to\infty} k^{1-\sigma} \left(q^{1/\sigma} k\right)^{\sigma} e^{-\rho t} \\
&= \lim\nolimits_{t\to\infty} (\sigma/\rho)^{\sigma} k^{1-\sigma} e^{-\rho t} \\
&= 0. \text{ (Lemma 2)}
\end{aligned}$$

■

Appendix C

In this appendix, we solve the government problem of optimal taxation policy in our special example under Chamley framework.

$$\max \int_0^\infty e^{-\rho t} \ln(\rho a)\, dt$$

$$\begin{aligned}
\text{subject to } \dot{a} &= (1-\tau_k)\alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} a - \rho a \\
\dot{b} &= g - \tau_k \alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} a \\
&\quad -\tau_l(1-\alpha)A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha}(a-b) \\
&\quad +\alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} b \\
\tau_k &\leq 1 \\
&\quad a_0 \text{ and } b_0 \text{ given}
\end{aligned}$$

Let λ and μ be the co-state variables on a and b respectively. Let ν be the Lagrangian multiplier on $\tau_k \leq 1$. The first order conditions are

$$-(\lambda+\mu)\alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} a + \nu = 0 \tag{48}$$

$$\begin{aligned}
&-\lambda(1-\tau_k)\alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} a\frac{1-\alpha}{\alpha(1-\tau_l)} \\
&+\mu\tau_k\alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} a\frac{1-\alpha}{\alpha(1-\tau_l)} \\
&-\mu(1-\alpha)A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha}(a-b) \\
&+\mu\tau_l(1-\alpha)A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha}(a-b)\frac{1-\alpha}{\alpha(1-\tau_l)} \\
&-\mu\alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} b\frac{1-\alpha}{\alpha(1-\tau_l)} = 0
\end{aligned} \tag{49}$$

$$\nu(1-\tau_k) = 0 \tag{50}$$

$$
\begin{aligned}
\dot{\lambda} &= \rho\lambda - \frac{1}{a} - \lambda\left[(1-\tau_k)\alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} - \rho\right] \\
&\quad +\mu\left[\tau_k\alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} + \tau_l(1-\alpha)A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha}\right]
\end{aligned}
$$

$$
\dot{\mu} = \rho\mu - \mu\tau_l(1-\alpha)A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} - \mu\alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha}
$$

The initial boundary conditions and the transversality conditions are

$$
\begin{aligned}
&a_0 \text{ and } b_0 \text{ given} \\
a\lambda e^{-\rho t} &\rightarrow 0 \text{ and } b\mu e^{-\rho t} \rightarrow 0
\end{aligned}
$$

From the government's objective function, we see that it is not optimal to have $a = 0$ at any moment. Thus $a \neq 0$ is always true.

Also, the constraint $\tau_k \leq 1$ can not be binding forever. Let T be the time after which the constraint is not binding ($T = 0$ is not ruled out at this moment).

Consider what happens when $t \geq T$. By definition of T, we know that $\nu(t) = 0$. Hence from Eq. (48) and the fact that $a \neq 0$, we have

$$
\lambda + \mu = 0
$$

And from (49) and the fact that $k = a - b > 0$, we have

$$
\tau_l \equiv 0
$$

Combining the differential equations on λ and μ and using the fact that $\lambda + \mu = 0$, we obtain,

$$
a\lambda = \frac{1}{\rho}
$$

As a result,

$$
\dot{\lambda} = \lambda\left[\rho - \alpha A\left[(1-\alpha)A\right]^{(1-\alpha)/\alpha}\right]
$$

$$
\begin{aligned}
\frac{\dot{a}}{a} + \frac{\dot{\lambda}}{\lambda} &= (1-\tau_k)\alpha A\left[(1-\alpha)A(1-\tau_l)\right]^{(1-\alpha)/\alpha} - \rho + \rho - \alpha A\left[(1-\alpha)A\right]^{(1-\alpha)/\alpha} \\
&= -\tau_k\alpha A\left[(1-\alpha)A\right]^{(1-\alpha)/\alpha}
\end{aligned}
$$

Since $a\lambda = 1/\rho$ implies that the LHS of the above equation is zero, it must be true that $\tau_k = 0$, for $t \geq T$. Therefore whenever $\tau_k < 1$, τ_k and τ_l must be zero. In order for this to be possible, then we must need the government to own enough private assets at time T so that the returns on the assets are just sufficient to cover its spending thereafter. In other words, $\dot{b} = 0$ after T. Thus from $\dot{b}$ equation, we obtain:

$$b(T) = -\frac{g}{\alpha A\left[(1-\alpha)A\right]^{(1-\alpha)/\alpha}} \tag{51}$$

and $b(t)$ will stay at that level for any $t > T$.

When $t < T$, $\tau_k = 1$ by the definition of T. Eq. (49) can be simplified and rewritten as follows:

$$\mu\tau_l = 0 \tag{52}$$

Note that $-\mu$ has the interpretation of marginal tax excess burden so that μ is always negative[5]. Hence,

$$\tau_l = 0 \tag{53}$$

Substitute $\tau_k = 1$ and $\tau_l = 0$ to the differential equations on a and b, we find that for $t < T$,

$$\dot{a} = -\rho a$$

$$\dot{b} = g - \alpha A\left[(1-\alpha)A\right]^{(1-\alpha)/\alpha} a + \alpha A\left[(1-\alpha)A\right]^{(1-\alpha)/\alpha} b$$

Note that the constant, $\alpha A\left[(1-\alpha)A\right]^{(1-\alpha)/\alpha}$, is the pre-tax real interest rate r, we can simplify and re-arrange the equation above and then multiply both

[5]If the government starts with a b_0 (negative) which generates interest revenue *more than sufficient* to cover its spending, then $\mu \equiv 0$. As in Chamley [8], we assume that it is unlikely for the government to have large negative b at time zero. Therefore μ is strictly negative. Note that $\mu(t)$ remains to be negative even after time T when its interest revenue is *just sufficient* to cover its spending. The reason is that any marginal increase of government spending will activate the use of distortionary taxes and the distortion is measured by the excess burden $-\mu$.

sides by e^{-rt}:

$$\left[\dot{b} - rb\right] e^{-rt} = ge^{-rt} - ra_0 e^{-\rho t - rt}$$

Take integral from 0 to T,

$$b_T e^{-rT} - b_0 = g\frac{1 - e^{-rT}}{r} - \frac{ra_0}{\rho + r}\left[1 - e^{-(\rho+r)T}\right]$$

Substituting in $b_T = -g/r$ from Eq. (51), we get an equation that determines T:

$$\frac{ra_0}{\rho + r}\left[1 - e^{-(\rho+r)T}\right] = b_0 + \frac{g}{r}$$

Thus, the optimal tax policy is:

$$\begin{aligned} \tau_k &= \begin{cases} 1 \text{ when } t < T \\ 0 \text{ when } t \geq T \end{cases} \\ \tau_l &= 0 \text{ for all } t \end{aligned}$$

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A List of Symbols

α	alpha
τ	tau
k_α	kay subscript alpha
$k_{\alpha 0}$	kay subscript alpha zero
ρ	rho
λ	lambda
ξ	xi
ξ_0	xi subscript zero
σ	sigma
Ω	uppercase omega
γ	gamma
θ	theta
τ_k	tau subscript kay
τ_l	tau subscript el
μ	mu
ν	nu
$\int$	integral
∞	infinity